

The maximal principle for properly immersed submanifolds and its applications

Yong Luo

Abstract

In this note we consider the Liouville type theorem for a properly immersed submanifold M in a complete Riemannian manifold N . Assume that the sectional curvature K^N of N satisfies $K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)^{\frac{\alpha}{2}}$ for some $L > 0, 2 > \alpha \geq 0$ and $q_0 \in N$.

(i) If $\Delta|\vec{H}|^{2p-2} \geq k|\vec{H}|^{2p}$ ($p > 1$) for some constant $k > 0$, then we prove that M is minimal.

(ii) Let u be a smooth nonnegative function on M satisfying $\Delta u \geq ku^a$ for some constant $k > 0$ and $a > 1$. If $|\vec{H}| \leq C(1 + \text{dist}_N(\cdot, q_0)^2)^{\frac{\beta}{2}}$ for some $C > 0, 0 \leq \beta < 1$, then $u = 0$ on M .

As applications we get some nonexistence result for p -biharmonic submanifolds.

1 Introduction

In the past several decades harmonic maps play a central role in geometry and analysis. Let $\phi : (M^m, g) \rightarrow (N^{m+t}, h)$ be a map between Riemannian manifolds (M, g) and (N, h) . The energy of ϕ is defined by

$$E(\phi) = \int_M \frac{|d\phi|^2}{2} d\nu_g,$$

where $d\nu_g$ is the volume element on (M, g) .

The Euler-Lagrange equation of E is

$$\tau(\phi) = \sum_{i=1}^m \{\tilde{\nabla}_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)\} = 0,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on the pullback bundle $\phi^{-1}TN$ and $\{e_i\}$ is a local orthonormal frame field on M . In 1983, Eells and Lemaire [17] proposed

to consider the bienergy functional

$$E_2(\phi) = \int_M \frac{|\tau(\phi)|^2}{2} d\nu_g,$$

where $\tau(\phi)$ is the tension field of ϕ . Recall that ϕ is harmonic if $\tau(\phi) = 0$. The Euler-Lagrange equation for E_2 is

$$\tau_2(\phi) = \tilde{\Delta}(\tau(\phi)) - \sum_{i=1}^m R^N(\tau(\phi), d\phi(e_i))d\phi(e_i) = 0.$$

To further generalize the notion of harmonic maps, Peter and Moser[21](see also [20]) considered the $p(p > 1)$ -bienergy functional as follows:

$$E_p(\phi) = \int_M |\tau(\phi)|^p d\nu_g.$$

The p -bitension field $\tau_p(\phi)$ is

$$\tau_p(\phi) = \tilde{\Delta}(|\tau(\phi)|^{p-2}\tau(\phi)) - \sum_{i=1}^m \left(R^N(|\tau(\phi)|^{p-2}\tau(\phi), d\phi(e_i))d\phi(e_i) \right). \quad (1.1)$$

The Euler-Lagrange equation for E_p is $\tau_p(\phi) = 0$ and a map u satisfying $\tau_p(\phi) = 0$ is called p -biharmonic maps. If $\phi : (M^m, g) \rightarrow (N^{m+t}, h)$ is an isometry immersion, then we call u p -biharmonic submanifold and 2-biharmonic submanifolds are called biharmonic submanifolds.

For biharmonic submanifolds, we have the well known Chen's conjecture [9]:

conjecture 1.1. *Every biharmonic submanifold in \mathbf{E}^n is minimal.*

Chen's conjecture inspires the research on the nonexistence of biharmonic submanifolds in nonpositively curved manifolds ([1][2][5] [7][8][9] [10][11] [15][16][18][22] [23][25][26][27][28] [29][30] [31] [32] [33] etc.). Motivated by Chen's conjecture, Yingbo Han [19] proposed the following conjecture:

conjecture 1.2. *Every complete p -biharmonic submanifolds in nonpositively curved Riemannian manifold is minimal.*

Some partial affirmative answers to conjecture 1.2 were proved in [19] and [6]. In this note we will continue to consider the nonexistence of p -biharmonic submanifolds in nonpositively curved Riemannian manifold. Before mentioning our main result, we define the following notion(see [27]).

Definition 1.1. For a complete manifold (N, h) and $\alpha \geq 0$, if the sectional curvature K^N of N satisfies

$$K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)^{\frac{\alpha}{2}},$$

for some $L > 0$ and $q_0 \in M$, then we say that K^N has a polynomial growth bound of order α from below.

We have

Theorem 1.1. *Let (M, g) be a properly immersed submanifold in a complete Riemannian manifold (N, h) whose sectional curvature K^N has a polynomial growth bound of order less than 2 from below. Assume that there exists a positive constant $k > 0$ such that $(p > 1)$*

$$\Delta |\vec{H}|^{2p-2} \geq k |\vec{H}|^{2p} \text{ on } M. \quad (1.2)$$

Then M is minimal.

Remark 1.1. *When $p = 2$, theorem 1.1 was proved by Maeta (see [27]). Our proof follows his argument by using the second derivatives' test to our new test functions. Maeta's argument was developed by Cheng and Yau in the 1970s (see [12][13][14] etc.).*

Theorem 1.1 implies the following nonexistence result of p -biharmonic submanifolds.

Theorem 1.2. *Let (M, g) be a properly immersed p -biharmonic submanifold in a complete nonpositively curved Riemannian manifold (N, h) whose sectional curvature K^N has a polynomial growth bound of order less than 2 from below, then M is minimal.*

Using the same argument, we also have the following Liouville type theorem.

Theorem 1.3. *Let (M, g) be a properly immersed submanifold in a complete Riemannian manifold (N, h) whose sectional curvature K^N has a polynomial growth bound of order less than 2 from below. Assume that u is a smooth nonnegative function on M satisfying*

$$\Delta u \geq k u^a \text{ on } M, \quad (1.3)$$

where $k > 0, a > 1$ are constants. If $|\vec{H}| \leq C(1 + \text{dist}_N(\cdot, q_0)^2)^{\frac{\beta}{2}}$ for some $C > 0, 0 \leq \beta < 1$ and $q_0 \in N$, then $u = 0$ on M . Here \vec{H} is the mean curvature vector field of M in N .

This Liouville type theorem was first found by Maeta. In [27] he proved the case of $a = 2$.

The rest of this paper is organized as follows: In section 2 we will briefly recall the theory of p -biharmonic submanifolds and submanifold theory. Our main theorems are proved in section 3.

2 Preliminaries

In this section we give more details on the definitions of harmonic maps, biharmonic maps, p -biharmonic maps and p -biharmonic submanifolds.

Let $u : (M^m, g) \rightarrow (N^{m+t}, h)$ be a map from an m -dimensional Riemannian manifold (M, g) to an $m + t$ -dimensional Riemannian manifold (N, h) . The energy of u is defined by

$$E(u) = \int_M \frac{|du|^2}{2} d\nu_g.$$

The Euler-Lagrange equation of E is

$$\tau(u) = \sum_{i=1}^m \{ \tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i) \} = 0,$$

where we denote ∇ the Levi-Civita connection on (M, g) , and $\tilde{\nabla}$ the induced Levi-Civita connection of the pullback bundle $u^{-1}TN$. A map $u : (M^m, g) \rightarrow (N^{m+t}, h)$ is called a harmonic map if $\tau(u) = 0$. To generalize the notion of harmonic maps, Eells and Lemaire [17] proposed to consider the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} d\nu_g.$$

The Euler-Lagrange equation for E_2 is (see [24])

$$\tau_2(u) = \tilde{\Delta}(\tau(u)) - \sum_{i=1}^m R^N(\tau(u), du(e_i)) du(e_i) = 0.$$

To further generalize the notion of harmonic maps, Han and Feng [20] (see also [21]) introduced the F -bienergy functional

$$E_F(u) = \int_M F\left(\frac{|\tau(u)|^2}{2}\right) d\nu_g,$$

where $F : [0, +\infty)$ and $F'(x) > 0$ if $x > 0$.

The critical points of the F -bienergy functional with $F(x) = (2x)^{\frac{p}{2}}$ ($p > 1$) are called p -biharmonic maps and isometric p -biharmonic maps are called p -biharmonic submanifolds.

The p -bitension field $\tau_p(u)$ is

$$\tau_p(u) = \tilde{\Delta}(|\tau(u)|^{p-2} \tau(u)) - \sum_{i=1}^m \left(R^N(|\tau(u)|^{p-2} \tau(u), du(e_i)) du(e_i) \right). \quad (2.1)$$

A p -biharmonic map satisfies $\tau_p(u) = 0$.

Now we briefly recall the submanifold theory. Let $u : (M, g) \rightarrow (N, h)$ be an isometric immersion from an m -dimensional Riemannian manifold into an $m + t$ -dimensional Riemannian manifold. The second fundamental form $B : TM \times TM \rightarrow T^\perp(M)$ is defined by:

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y), X, Y \in \Gamma(TM), \quad (2.2)$$

where $\bar{\nabla}$ is the Levi-Civita connection on N and ∇ is the Levi-Civita connection on M . The Weingarten formula is given by

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi, X \in \Gamma(TM), \quad (2.3)$$

where A_ξ is called the Weingarten map w.r.t. $\xi \in T^\perp(M)$, and ∇^\perp denotes the normal connection on the normal bundle of M in N . For any $x \in M$, the mean curvature vector field \vec{H} of M at x is

$$\vec{H} = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i).$$

If u is an isometry immersion, we see that $\{du(e_i)\}$ is a local orthonormal frame of M . In addition, for any $X, Y \in \Gamma(TM)$,

$$\nabla du(X, Y) = \tilde{\nabla}_X(du(Y)) - du(\nabla_X^\perp Y) = B(X, Y), \quad (2.4)$$

where $\tilde{\nabla}$ is the connection on the pull back bundle $u^{-1}TN$, whose fiber at a point $x \in M$ is $T_{u(x)}N = T^\perp M \oplus TM$. Therefore if u is an isometric immersion,

$$\tau(u) = \text{tr} \nabla du = \text{tr} B = m \vec{H},$$

and a p -biharmonic submanifold satisfies the following equation:

$$\tau_p(u) = \tilde{\Delta}(|\vec{H}|^{p-2} \vec{H}) - \sum_{i=1}^m \left(R^N(|\vec{H}|^{p-2} \vec{H}, e_i) e_i \right) \quad (2.5)$$

where $\tilde{\Delta} = \sum_{i=1}^m (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i})$, $\tilde{\nabla}$ is the connection on the pullback bundle, and R^N is the Riemannian curvature tensor on N .

From (2.3), we get for any vector field $\xi \in \Gamma(T^\perp M)$:

$$\begin{aligned} \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \xi &= \tilde{\nabla}_{e_i} (\nabla_{e_i}^\perp \xi - A_\xi e_i) \\ &= \nabla_{e_i}^\perp \nabla_{e_i}^\perp \xi - \tilde{\nabla}_{e_i} A_\xi e_i - A_{\nabla_{e_i}^\perp \xi} e_i \\ &= \nabla_{e_i}^\perp \nabla_{e_i}^\perp \xi - \nabla_{e_i} A_\xi e_i - B(e_i, A_\xi e_i) + A_{\nabla_{e_i}^\perp \xi}(e_i), \end{aligned}$$

and

$$\begin{aligned} &\tilde{\nabla}_{\nabla_{e_i} e_i} \xi \\ &= \nabla_{\nabla_{e_i} e_i}^\perp \xi - A_\xi(\nabla_{e_i} e_i). \end{aligned}$$

Combining the above two identities, we get

$$\begin{aligned}
\tilde{\Delta}\xi &= \nabla_{e_i}^\perp \nabla_{e_i}^\perp \xi - \nabla_{e_i} A_\xi e_i - B(e_i, A_\xi e_i) + A_{\nabla_{e_i}^\perp \xi}(e_i) \\
&+ \nabla_{\nabla_{e_i}^\perp \xi}^\perp \xi - A_\xi(\nabla_{e_i} e_i) \\
&= \Delta^\perp \xi - \nabla_{e_i} A_\xi e_i - A_\xi(\nabla_{e_i} e_i) - B(e_i, A_\xi e_i) + A_{\nabla_{e_i}^\perp \xi}(e_i)
\end{aligned}$$

Therefore by decomposing the p -biharmonic submanifold equation into its normal and tangential parts respectively we get [19]:

$$\Delta^\perp \left(|\vec{H}|^{p-2} \vec{H} \right) - \sum_{i=1}^m B(A_{|\vec{H}|^{p-2} \vec{H}} e_i, e_i) + \sum_{i=1}^m (R^N(|\vec{H}|^{p-2} \vec{H}, e_i) e_i)^\perp = 0, \quad (2.6)$$

$$Tr_g(\nabla A_{|\vec{H}|^{p-2} \vec{H}}) + Tr_g[A_{\nabla^\perp |\vec{H}|^{p-2} \vec{H}}(\cdot)] - \sum_{i=1}^m (R^N(|\vec{H}|^{p-2} \vec{H}, e_i) e_i)^\top = 0. \quad (2.7)$$

3 Proof of theorems

In this section, we will need the following Hessian comparison theorem(see [4]).

Lemma 3.1. *Let (N, h) be a complete Riemannian manifold with $\text{sect} \geq K$ ($K < 0$). For any point $q \in M$ the distance function $r(x) = d(x, q)$ satisfies*

$$D^2 r \leq \sqrt{|K|} \coth(\sqrt{|K|} r) h,$$

at all points where r is smooth (i.e. away from q and the cut locus). Here $D^2 r$ denotes the Hessian of r .

3.1 Proof of Theorem 1.1

Proof. If M is compact we see that $\vec{H} = 0$ follows from the standard maximal principle. Therefore we assume that M is noncompact. We will prove the theorem by a contradiction argument. Here we follow Maeta's([27]) argument by choosing new test functions.

Suppose that $\vec{H}(x_0) \neq 0$ for some $x_0 \in M$. Set $u(x) = |\vec{H}(x)|^{2p-2}$ for $x \in M$. For each $\rho > 0$ let

$$F(x) = F_\rho(x) = (\rho^2 - r^2(\phi(x)))^{2p-2} u(x),$$

for $x \in M \cap X^{-1}(\bar{B}_\rho)$, where $\phi : M \rightarrow R^n$ is the isometric immersion, B_ρ is the standard ball in R^n with radius ρ and $r(\phi(x)) = \text{dist}_N(\phi(x), q_0)$ for some $q_0 \in N$.

Assume that $x_0 \in X^{-1}(B_{\rho_0})$. For each $\rho \geq \rho_0$, $F = F_\rho$ is a nonnegative function which is not identically zero on $M \cap X^{-1}(\bar{B}_\rho)$ and equals zero on the boundary. Assume that $q \in M \cap X^{-1}(B_\rho)$ is the maximum point of $F(q)$ exists because ϕ is properly immersed).

(i) $\phi(q)$ is not on the cut loss of q_0 . Then $\nabla F(q) = 0$ and hence we get at q

$$\frac{\nabla u}{u} = \frac{(2p-2)\nabla r^2(\phi(x))}{\rho^2 - r^2(\phi(x))}. \quad (3.1)$$

In addition at q

$$\begin{aligned} 0 \geq \Delta F(x) &= (2p-2)(2p-3)(\rho^2 - r^2(\phi(x)))^{2p-4} |\nabla r^2(\phi(x))|^2 u(x) \\ &\quad - (2p-2)(\rho^2 - r^2(\phi(x)))^{2p-3} \Delta r^2(\phi(x)) u(x) \\ &\quad - 2(2p-2)(\rho^2 - r^2(\phi(x)))^{2p-3} \langle \nabla r^2(\phi(x)), \nabla u \rangle_g \\ &\quad + (\rho^2 - r^2(\phi(x)))^{2p-2} \Delta u. \end{aligned} \quad (3.2)$$

Combining inequalities (3.1) and (3.2) we have at q

$$\frac{\Delta u(x)}{u(x)} \leq \frac{(2p-2)(2p-1)|\nabla r^2(\phi(x))|^2}{(\rho^2 - r^2(\phi(x)))^2} + \frac{(2p-2)\Delta r^2(\phi(x))}{\rho^2 - r^2(\phi(x))}. \quad (3.3)$$

By a direct computation we see that

$$|\nabla r^2(\phi(x))|_g^2 \leq 4mr^2(\phi(x)),$$

and

$$\begin{aligned} \Delta r^2(\phi(x)) &= 2 \sum_{i=1}^m \langle (\bar{\nabla} r)(\phi(x)), d\phi(e_i) \rangle^2 \\ &\quad + 2r(\phi(x)) \sum_{i=1}^m (D^2 r)(\phi(x)) \langle d\phi(e_i), d\phi(e_i) \rangle + 2r(\phi(x)) \langle (\bar{\nabla} r)(\phi(x)), \tau(\phi)(x) \rangle \\ &\leq 2m + 2r(\phi(x)) \sum_{i=1}^m (D^2 r)(\phi(x)) \langle d\phi(e_i), d\phi(e_i) \rangle + 2mr(\phi(x)) |\vec{H}(x)|, \end{aligned} \quad (3.4)$$

where $m = \dim M$, $\bar{\nabla}$ is the gradient on (N, h) and $D^2 r$ denotes the Hessian of r . Since the sectional curvature K^N of N satisfies $K^N \geq -L(1 + r^2)^{\frac{\alpha}{2}}$, by the Hessian comparison theorem (see lemma 3.1) we get

$$\sum_{i=1}^m (D^2 r)(\phi(x)) \langle d\phi(e_i), d\phi(e_i) \rangle \leq m \sqrt{L(1 + r^2)^{\frac{\alpha}{2}}} \coth \left(\sqrt{L(1 + r^2)^{\frac{\alpha}{2}}} r(\phi(x)) \right). \quad (3.5)$$

Combining the last two inequalities we obtain

$$\begin{aligned} \Delta r^2(\phi(x)) &\leq 2m + 2m \sqrt{L(1 + r^2)^{\frac{\alpha}{2}}} r(\phi(x)) \coth \left(\sqrt{L(1 + r^2)^{\frac{\alpha}{2}}} r(\phi(x)) \right) \\ &\quad + 2mr(\phi(x)) |\vec{H}(x)|. \end{aligned} \quad (3.6)$$

Recall that $\Delta|\vec{H}|^{2p-2} \geq k|\vec{H}|^{2p}$, i.e. $\Delta u \geq ku^{\frac{2p}{2p-2}}$, thus from inequalities (3.3), (3.6) we obtain

$$\begin{aligned} ku(q)^{\frac{1}{p-1}} &\leq \frac{4m(2p-2)(2p-1)r^2(\phi(q))}{(\rho^2 - r^2(\phi(q)))^2} \\ &+ \frac{(2p-2)\left\{2m + 2m\sqrt{L(1+r^2)^{\frac{\alpha}{2}}}r(\phi(q))\coth\left(\sqrt{L(1+r^2)^{\frac{\alpha}{2}}}r(\phi(q))\right)\right\}}{\rho^2 - r^2(\phi(q))} \\ &+ \frac{(2p-2)2mr(\phi(q))|\vec{H}(q)|}{\rho^2 - r^2(\phi(q))}. \end{aligned} \quad (3.7)$$

From the last inequality one gets

$$\begin{aligned} u(q) &\leq C(p, k, m)\left[\frac{r^{2p-2}(\phi(q))}{(\rho^2 - r^2(\phi(q)))^{2p-2}}\right. \\ &+ \frac{\left\{1 + \sqrt{L(1+r^2)^{\frac{\alpha}{2}}}r(\phi(q))\coth\left(\sqrt{L(1+r^2)^{\frac{\alpha}{2}}}r(\phi(q))\right)\right\}^{p-1}}{(\rho^2 - r^2(\phi(q)))^{p-1}} \\ &+ \left.\sqrt{u(q)r(\phi(q))}^{p-1}\frac{1}{(\rho^2 - r^2(\phi(q)))^{p-1}}\right], \end{aligned} \quad (3.8)$$

where $C(p, k, m)$ is a constant depends only on p, k, m . Therefore

$$\begin{aligned} F(q) &\leq C(p, k, m)[r^{2p-2}(\phi(q)) + \\ &\left\{1 + \sqrt{L(1+r^2)^{\frac{\alpha}{2}}}r(\phi(q))\coth\left(\sqrt{L(1+r^2)^{\frac{\alpha}{2}}}r(\phi(q))\right)\right\}^{p-1}(\rho^2 - r^2(\phi(q)))^{p-1} \\ &+ \sqrt{F(q)r(\phi(q))}^{p-1}] \end{aligned} \quad (3.9)$$

which implies that

$$F(q) \leq C(p, k, m, L)(1 + \rho^2)^{\frac{(\alpha+6)}{4}(p-1)},$$

where $C(p, k, m, L)$ is a constant depends only on p, k, m, L . Since q is the maximum of F , for any $x \in M \cap B_\rho$ we have

$$F(x) \leq F(q) \leq C(p, k, m, L)(1 + \rho^2)^{\frac{(\alpha+6)}{4}(p-1)}.$$

Therefore

$$|\vec{H}(x)|^{2p-2} \leq \frac{C(p, k, m, L)(1 + \rho^2)^{\frac{(\alpha+6)}{4}(p-1)}}{(\rho^2 - r^2(\phi(x)))^{2p-2}}, \quad (3.10)$$

for any $x \in M \cap B_\rho$ and $\rho \geq \rho_0$.

(ii) If $\phi(q)$ is on the cut loss of q_0 , then we use a method of Calabi (see [3]). Let σ be a minimal geodesic joining $\phi(q)$ and q_0 . Then for any q' in the

interior of σ , q' is not conjugate to q_0 . Fix for such a point q' . Let $U_{q'} \subseteq B_\rho$ be a conical neighborhood of the geodesic segment of σ joining q' and $\phi(q)$ such that for any $\phi(x) \in U_{q'}$, there is at most one minimizing geodesic joining q' and $\phi(x)$. Let $\bar{r}(\phi(x)) = \text{dist}_{U_{q'}}(\phi(x), q')$ in the manifold $U_{q'}$. Then we have $\bar{r}(\phi(x)) \geq \text{dist}_N(\phi(x), q')$, $r(\phi(x)) \leq r(q') + \bar{r}(\phi(x))$, $r(\phi(q)) = r(q') + \bar{r}(\phi(q))$. We claim that the function

$$F_{\rho, q'}(x) := (\rho^2 - \{r(q') + \bar{r}(\phi(x))\}^2)^{2p-2} u(x) \text{ for } x \in \phi^{-1}(U_{q'})$$

also attains a local maximum at the point q . In fact, for any point $x \in \phi^{-1}(U_{q'})$ we have

$$\begin{aligned} F_{\rho, q'}(q) &= (\rho^2 - \{r(q') + \bar{r}(\phi(q))\}^2)^{2p-2} u(q) \\ &= (\rho^2 - r^2(\phi(q)))^{2p-2} u(q) \\ &= F_\rho(q) \geq F_\rho(x) \\ &= (\rho^2 - r^2(\phi(x)))^{2p-2} u(x) \\ &\geq (\rho^2 - \{r(q') + \bar{r}(\phi(x))\}^2)^{2p-2} u(x) \\ &= F_{\rho, q'}(x). \end{aligned}$$

Therefore the claim is proved and we play the second derivative's test to $F_{\rho, q'}(x)$ at q , the same argument as before shows that

$$\begin{aligned} F_{\rho, q'}(q) &\leq C(p, k, m)[r^{2p-2}(\phi(q)) + \\ &\quad \left\{ 1 + \sqrt{L(1+r^2)^{\frac{\alpha}{2}}} r(\phi(q)) \coth\left(\sqrt{L(1+r^2)^{\frac{\alpha}{2}}} r(\phi(q))\right) \right\}^{p-1} (\rho^2 - r^2(\phi(q)))^{p-1} \\ &\quad + \sqrt{F_{\rho, q'}(q)} r(\phi(q))^{p-1}], \end{aligned}$$

which implies that

$$F_{\rho, q'}(q) \leq C(p, k, m, L)(1 + \rho^2)^{\frac{(\alpha+6)}{4}(p-1)}.$$

Take $q' \rightarrow q_0$ we have $F_{\rho, q'}(q) = F_\rho(q)$ and hence

$$F_\rho(q) \leq C(p, k, m, L)(1 + \rho^2)^{\frac{(\alpha+6)}{4}(p-1)}.$$

Therefore

$$|\vec{H}(x)|^{2p-2} \leq \frac{C(p, k, m)(1 + \rho^2)^{\frac{(\alpha+6)}{4}(p-1)}}{(\rho^2 - r^2(\phi(x)))^{2p-2}}, \quad (3.11)$$

for any $x \in M \cap B_\rho$ and $\rho \geq \rho_0$. Let $x = x_0$ and $\rho \rightarrow +\infty$ we get $\vec{H}(x_0) = 0$, a contradiction. Therefore M is minimal. \square

3.2 Proof of Theorem 1.2

Proof. Recall that the normal part of the p -biharmonic submanifolds is

$$\Delta^\perp \left(|\vec{H}|^{p-2} \vec{H} \right) - \sum_{i=1}^m B(A_{|\vec{H}|^{p-2} \vec{H}} e_i, e_i) + \sum_{i=1}^m (R^N(|\vec{H}|^{p-2} \vec{H}, e_i) e_i)^\perp = 0.$$

Therefore

$$\begin{aligned} \Delta |\vec{H}|^{2p-2} &= 2 \langle \Delta^\perp(|\vec{H}|^{p-2} \vec{H}), |\vec{H}|^{p-2} \vec{H} \rangle + 2 |\nabla(|\vec{H}|^{p-2} \vec{H})|^2 \\ &\geq 2 \sum_{i=1}^m \langle B(A_{|\vec{H}|^{p-2} \vec{H}} e_i, e_i), |\vec{H}|^{p-2} \vec{H} \rangle \\ &= 2 |\vec{H}|^{2p-4} \langle A_{\vec{H}} e_i, A_{\vec{H}} e_i \rangle \\ &\geq 2m |\vec{H}|^{2p}, \end{aligned}$$

where in the first inequality we used the assumption of nonpositive curvature. Therefore M is minimal by theorem 1.1. \square

3.3 Proof of Theorem 1.3

Proof. Similar to the proof of theorem 1.1 set $F_\rho(x) = (\rho^2 - r^2(\phi(x)))^{2a-2} u(x)$. If $u(x_0) \neq 0$, then using the second derivatives' test to F_ρ at the maximum point q for ρ big enough such that $x_0 \in B_\rho$, we will get

$$\begin{aligned} ku(q)^{\frac{1}{a-1}} &\leq \frac{4m(2a-2)(2a-1)r^2(\phi(q))}{(\rho^2 - r^2(\phi(q)))^2} \\ &+ \frac{(2a-2) \left\{ 2m + 2m \sqrt{L(1+r^2)^{\frac{a}{2}}} r(\phi(q)) \coth \left(\sqrt{L(1+r^2)^{\frac{a}{2}}} r(\phi(q)) \right) \right\}}{\rho^2 - r^2(\phi(q))} \\ &+ \frac{(2a-2)2mr(\phi(q))|\vec{H}(q)|}{\rho^2 - r^2(\phi(q))}. \end{aligned} \quad (3.12)$$

Therefore

$$\begin{aligned} u(q) &\leq C(a, k, m) \left[\frac{r^{2a-2}(\phi(q))}{(\rho^2 - r^2(\phi(q)))^{2a-2}} \right. \\ &+ \frac{\left\{ 1 + \sqrt{L(1+r^2)^{\frac{a}{2}}} r(\phi(q)) \coth \left(\sqrt{L(1+r^2)^{\frac{a}{2}}} r(\phi(q)) \right) \right\}^{a-1}}{(\rho^2 - r^2(\phi(q)))^{a-1}} \\ &\left. + |\vec{H}|^{a-1} r(\phi(q))^{a-1} \frac{1}{(\rho^2 - r^2(\phi(q)))^{a-1}} \right], \end{aligned} \quad (3.13)$$

which implies that

$$F_\rho(q) \leq C(a, k, m, L) \max\{(1 + \rho^2)^{\frac{\alpha+6}{4}}, (1 + \rho^2)^{\frac{\beta+3}{2}}\},$$

where we used the assumption that $|\vec{H}| \leq C(1 + \text{dist}_N(\cdot, q_0)^2)^{\frac{\beta}{2}}$. Therefore

$$(\rho^2 - r^2(\phi(x)))^{2a-2} u(x) \leq F_\rho(q) \leq C(a, k, m, L) \max\{(1 + \rho^2)^{\frac{\alpha+6}{4}(a-1)}, (1 + \rho^2)^{\frac{\beta+3}{2}(a-1)}\},$$

which implies that

$$u(x) \leq \frac{C(a, k, m, L) \max\{(1 + \rho^2)^{\frac{\alpha+6}{4}(a-1)}, (1 + \rho^2)^{\frac{\beta+3}{2}(a-1)}\}}{(\rho^2 - r^2(\phi(x)))^{2a-2}}.$$

Because $\alpha < 2$ and $\beta < 1$, let $x = x_0$ and $\rho \rightarrow +\infty$ we obtain $u(x_0) = 0$, a contradiction. Thus $u = 0$ on M . \square

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YONG LUO

SCHOOL OF MATHEMATICS AND STATISTICS,
 WUHAN UNIVERSITY, HUBEI 430072, CHINA
 yongluo@whu.edu.cn